

# **A random walk in reactor physics and neutron transport**

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The title of this paper alludes to two different meanings of “random”. First, the phrase “Random walk” refers to the fact that I selected, at random, a few topics which I myself found fascinating, surprising, and hence hopefully entertaining, in the hope that the reader will also find them entertaining. The phenomena that will be described and discussed here will reveal some unexpected features, which in some cases are puzzling or even counter-intuitive, and their explanation sometimes discloses commonly accepted misbeliefs or misunderstandings. I always found such cases very intriguing. Inevitably, such subjects do not constitute a continuous story, rather they are picked randomly, hence the first meaning of the phrase “random walk” in the title.

Curiosities similar to the types that will be discussed in this note are usually published as a “Letter to the Editor” or a “Technical Note”, since they do not contain new research results. A few examples are given in Ref [1] (meaning of the flux) and Refs [2] - [4] (number of collisions until slowing down). The readers are encouraged to check up these letters or technical notes. Many are, in contrast to the present article, quite short, often only one page, hence the “output/input ratio” in intellectual entertainment is quite high. I can also recommend the readers to watch out for such short notes by themselves (although, sadly, the number of such notes seems to be decreasing).

The second reason why the word “random” appears in the title is because the curious facts and phenomena which will be discussed here concern the randomness of neutron transport, manifesting itself in the fact that the number of neutrons in the system, or the number of detector counts during a time period, is a random number or random process (hence often referred to as neutron fluctuations or neutron noise). Random processes in general, whether about neutrons or other processes, have themselves fascinating and surprising properties. The subjects discussed in this small essay will hopefully also expedite a wider understanding of the properties and use of neutron fluctuations in nuclear systems.

With this introduction, I invite the reader to follow me on the random walk in the fascinating world of random particle transport.

## **The ubiquity and importance of fluctuations**

Before turning to reactor physics and neutron fluctuations, a few words about randomness and fluctuations in general. The randomness treated in this essay is not related to the intrinsically stochastic description of quantum mechanics of individual particle processes, where everything can only be formulated in terms of the so-called wave function, which evolves deterministically, but which only has a probabilistic interpretation. We will deal with macroscopic classical systems, which usually have a very large number of freedom (such as a system of  $10^{24}$  particles), where it is impossible to specify each variable with the desired accuracy, hence a statistical description is necessary. Thus, we cannot tell the outcome of the energy loss of neutrons in the laboratory system for an individual collision, only over a large number of collisions. In some cases quantum mechanics also plays a role, such as whether a reaction will be absorption or fission, and in the latter how many neutrons will be released. However, this fact will not require a quantum mechanical description, only the knowledge of the probabilities or probability distributions of the

corresponding processes and variables (such as the probability distribution of the number of neutrons in fission).

Since most systems in the macroscopic (hence observable) world are many-body systems, with some exaggerations one can say that everything in the world is random - but often we do not “see” the randomness (the fluctuations are too small). The fluctuations are invisible because the measurement does not have the proper resolution. Once we have a sufficiently sensitive equipment, we can see that most everyday processes (such as the temperature in a well-controlled isolated room) shows tiny, but definite fluctuations around the mean value.

The randomness, or the fluctuations, are often considered a nuisance, and this is expressed with the word “noise” which, in the world of acoustics, is surely a phenomenon that with the random distribution of its frequencies disturbs the enjoyment of music, consisting only of discrete frequencies. However, more often than not, through their rich information content, these fluctuations are very useful, and in biological processes even necessary for the existence of life.

My favourite example to support this last statement is the heart rate variability, which is based on the ECG (electrocardiogram) signals that can be easily measured. For a healthy person, they look periodical, such as shown in Fig. 1.

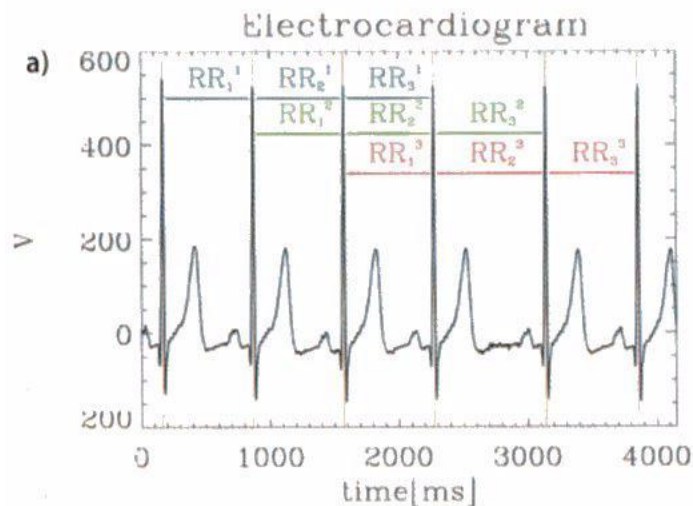


Fig. 1 Heartbeat signals

The heartbeats are characterized by the large peaks (denoted in medical praxis with the letter “R”), and the time between two beats, the “interbeat intervals” are denoted in the figure as RR<sub>1</sub>, RR<sub>2</sub> etc. Of course, a large variation of the interbeat intervals is an indication of heart disorder, but the opposite is not true “ad infinitum”. Namely, one would be tempted to think that for a person at total rest, these intervals are perfectly equal, and the healthier a person is, the more constant the interbeat intervals are. The “perfectly healthy” person in this concept would be one whose RR interval periods are exactly the same. However, a doctor would say that such a person is already “dead”, because the interbeat intervals *must* fluctuate with all living organisms. In the terminology of a lay mind, one could say that the interbeat intervals must vary such that the heart is “training” to be able to adjust the heart’s beat to meet a change in the physical load (running, lifting weights etc.).

Whatever the explanation of these fluctuations, plotting the RR intervals as a function of the beat number, one apparently gets a totally random process, seemingly without any structure (see the right hand side of Fig. 2).

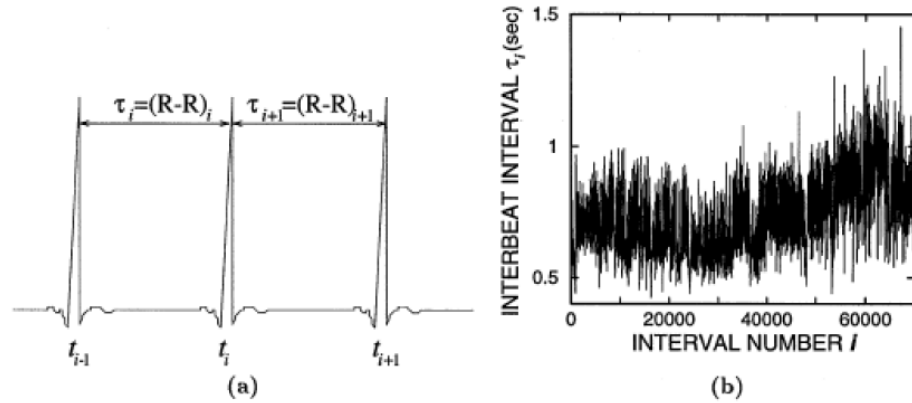


Fig. 2. Heartbeat signals (left) and a sequence of interbeat time intervals (right), showing a random behaviour

However, a wavelet analysis readily reveals a self-similar structure (see Fig. 3 to the right). The curve in the top is the variation of the interbeat signals, the lower two figures show their wavelet transforms, which indicates a self-similar structure: the lowermost figure is an enlarged view of the area in the red box in the middle figure, which has the same structure as the whole of the middle figure. Such a self-similar structure exhibits a fractal property, with a corresponding fractal dimension. Experience shows that for healthy persons, there is one single fractal dimension associated with the heartbeat data; however, the fractal structure of the interbeat intervals for persons with incipient heart failures changes, e.g. develops a bi-fractal structure, which can be used both for early detection of beginning heart failures, as well as to determine the type of heart failure. The analysis of heartbeat signals and their use of diagnostics is not very much unlike of that of analysing process data from nuclear reactors; so much so, that nuclear engineers have also been involved in analysis of medical data. At several nuclear engineering conferences there are sessions on analysis of medical signals. The present author has also endeavoured such a study [5].

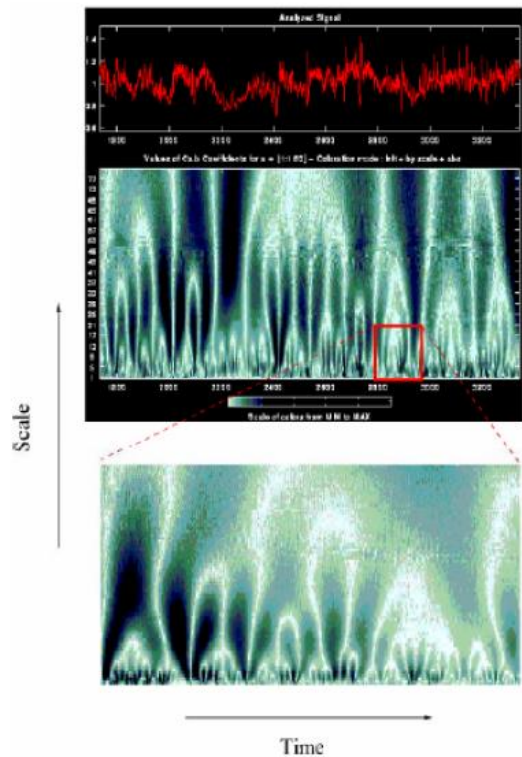


Fig.3 Interbeat intervals and their wavelet transform

One can easily find some intuitive illustrations of the use of the rich information content of random processes. Take as an example birds flying in flocks. The paths of the individual birds look disordered and random, as they constantly change directions, partly to avoid collisions with each other. But they succeed in this exceptionally well – birds in a flock do not collide with each other unintentionally. As a contrast, it often happens that birds are killed by collisions with cars. On first sight this might feel surprising; a car usually moves on a straight line with constant speed,

so one would think it is very easy to predict its trajectory and hence avoid it. But the bird brain does not work this way; the movement on a straight line with constant velocity has too little information. A “noisy” movement is much richer in information, and hence the brain of the birds processes it easier. Of course, in order to do that, they have to be able to interpret the random character of the movement; but apparently it better suits the brain to process a more complex information to make predictions than that of a simple process which is poor in information. Another example is the vision of a bird of prey hovering above the field. It will often discover a prey only when the prey is moving, even if very chaotically. A prey standing still often avoids being detected, because it sends too little information compared to that of a moving prey.

In what follows, we will see how the more complex character of neutron chains in a multiplying medium (compared to the emission of neutrons from a radioactive source) help us to extract information about the system, once we know how to interpret the increased complexity of the underlying physical process. Soon we will also encounter a puzzling observation, which shows the beautiful depths of the phenomenon, and that of random processes in general.

### **Family trees and neutron chains as branching processes**

Neutron chains and family trees have much in common what regards their random temporal evolution. They belong to the category that mathematicians call “branching processes”. The branching simply means that at random time points some events will occur, in which one entity will be converted into a random number of new entities (“descendants”). Here the probability of having at least two descendants must be larger than zero, otherwise the process is not branching. In this respect “branching” is synonymous with “multiplication”. On the other hand, the probability for a given event that there will be no descendants can also be larger than zero, which means that the process can also die out. For neutrons, the branching means fission, where one incoming neutron can induce the emission of a random number of new neutrons, zero also allowed (especially if absorption is also taken into account). In a family tree, the events are not associated with time, rather with generations, as will be later seen more in detail. The randomness of this process is expressed in the intuitively clear fact that even if the average of descendants is one, the process can still either grow or die out.

This shows that the statistical properties of a branching process are more involved than the case of neutron emission from a radioactive source. This latter has Poisson statistics, which is “boringly” simple; it is defined by one single parameter, the emission intensity, and hence all statistical moments (mean, variance etc.) carry the same information (e.g. the variance is equal to the mean). If we know the mean value, we know everything about the statistics of the process. This is a consequence of the fact that the individual events (particle emissions) are independent.

The reason for the more complicated, and hence more interesting statistics of a branching process is that its random variables, such as particle numbers, are not independent, rather they are correlated (their covariance, i.e. the joint probability of two variables minus the product of the probability of the two variables, is not zero). Here we already arrived to one of the rather subtle properties of branching processes which the reader might like to contemplate on. For a neutron chain, i.e. a collection of all generations started by a single neutron, in each individual fission, the number of new neutrons is independent of any other fissions in the system; and for the new neutrons born in fission, their fate is independent from each other in the sense that both the time for the next fission, as well as the number of new neutrons born in their respective fissions, are independent from each other. The independence of the development of the individual chains is actually a basic assumption which we set up for the equations that describe the evolution of the probability of the number of the neutrons in the system (so called master equations, or Chapman-Kolmogorov equations). Yet, due to the fact that they were born simultaneously, will incur

temporal correlations in the number of the neutrons at different times in the system. This is obtained from the solution of the master equations, set up for the evolution of the probability.

The apparent contradiction that the independence of the events in the process still leads to correlations in the numbers of the entities in the process might be illustrated qualitatively by stating that although two twins, born simultaneously, will die independently of each other, none of them will survive the other with a thousand years (or hundreds of generations). Their fates, although controlled by independent events, will strongly overlap. It is the common birth, the branching, which will create correlations between the number of descendants at a given time, despite that the individual life events are not affecting each other.

The significance of the correlations in the statistics of random processes, in comparison with that of the independent processes, obeying a pure Poisson process, can be described as follows. Positively correlated processes, i.e. when the presence of one entity indicates the increased likelihood of the presence of another entity (such as in a family tree or a neutron chain), have “over-Poisson” statistics, which simply means that the variance is larger than the mean (the relative variance is larger than unity). Random processes that are subject to a conservation law, have the property that the presence of one entity excludes or diminishes the probability of the presence of another entity, show negative correlations. Such processes have a sub-Poisson variance, i.e. the variance is smaller than the mean.

With this introduction we will now explore some properties of the branching processes by the story of the mathematical treatment of the extinction of family lines.

### **The extinction of family names**

The question of the extinction of the family names dates back to the early 19<sup>th</sup> century and is associated with the Reverend Henry William Watson, clergyman, mathematician and alpinist, and the mathematician Francis Galton. The story is very entertainingly described by D. G. Kendall in an essay presented at the 100<sup>th</sup> anniversary of the London Mathematical Society [6], as well as in the book of T. E. Harris [7]. Galton became interested in the observation by a certain M. Alphonse De Candolle (1806-1893) regarding the decay of famous families, “men of note”: peers, judges, and the like. De Candolle put forth the hypothesis, or question, whether the extinction of the family names of such noble families is caused by the fact that “*a rise in physical comfort and intellectual capacity is necessarily accompanied by diminution of fertility*”.

Galton did not like this conclusion (as I guess most of us having a profession requiring a certain intellectual capacity would not like either) and asked the help of Watson who formulated the problem in mathematical terms and gave a solution. Before getting to this, I have to attach the remark that some aspects of the formulation and interpretation of the problem reflect the standards of society as it was then, with a very different view on gender equality as it is today. For instance, the extinction of the family names is purely associated with the male line; an assumption that is not valid any longer.

A simplified version of Watson’s statement of the problem is then as follows:

“A large nation, of whom we will only concern ourselves with the adult males,  $N$  in number, and who each bear separate surnames, colonise a district. Their law of population is such that, in each generation,  $p_0$  per cent of the adult males have no male children who reach adult life;  $p_1$  have one such male child;  $p_2$  have two; and so on. Find (1) what proportion of the surnames will have become extinct after  $r$  generations; and (2) how many instances there will be of the same surname being held by  $m$  persons.” (An answer by Kendall, with an interesting side-line, will be given later).

A simpler task, which Watson solved, is that of the extinction probability  $q$ . That is, starting with one family, what is the probability, that when  $r$  goes to infinity, the family becomes extinct?

Instead of giving Watson's solution, which is somewhat complicated, we will quote here a simpler one, which was provided by Agner Krarup Erlang in 1929. He wrote down what we today would call a backward type master equation for the extinction probability  $q$ . He argued that this probability is equal to the probabilities of the mutually exclusive events that: either there will be no males in the first generation (with probability  $p_0$ ); *or* there will be one male descendant with probability  $p_1$ , which then has to die out (with the same extinction probability  $q$  that we are looking for); *or* there will be two male descendants with probability  $p_2$ , which then both have to die out; since they die out independently, this probability is equal to  $q^2$ ; and so on. This can be written as

$$q = p_0 + p_1 q + p_2 q^2 + \dots = \sum_{n=0}^{\infty} p_n q^n \quad (1)$$

The right hand side of this equation is actually the definition of the generation function of the probability distribution  $p_n$ . The generating function  $g(z)$  of a discrete probability distribution  $p_n$  is a function of the continuous variable  $z$ , which is defined as

$$g(z) = \sum_{n=0}^{\infty} p_n z^n \quad (2)$$

The generating function can be interpreted as a transform of the discrete probability distribution  $p_n$  into the continuous function  $g(z)$ . All generating functions have the property of  $g(1) = 1$ , which can be easily confirmed from Eq. (2). Further, from the generating function, the moments of the probability distribution can be easily obtained. For instance, the first moment (expectation) is obtained as

$$\langle n \rangle \equiv \nu = \sum_{n=0}^{\infty} n p_n = g'(z)|_{z=1} \quad (3)$$

With the concept of the generating function, the extinction equation, Eq. (1), can be written as

$$q = g(q) \quad (4)$$

This is a transcendental equation, since  $g(q)$  is a polynomial in  $q$ , whose order is equal to the maximum number of descendants which have a non-zero probability. However, a qualitative solution, which will actually give an exact quantitative solution with a graphical interpretation, shown in Fig. 4. The straight line (orange) at  $45^\circ$  represents the left hand side of Eq. (4), and the curve (blue) represents the right hand side of (4). The solution is the  $q$  value where the two lines intersect. Here we note that  $g(q)$  is always a convex function, since all coefficients in its Taylor series are positive (they are the probabilities  $p_n$ ). Since  $g(1) = 1$ , it follows that  $q = 1$  (extinction with 100% certainty) is always a solution of the extinction equation. Interestingly, Watson believed (erroneously) that this is the only solution, i.e. all families will always die out. We will see that this is not always the case, and is valid only to subcritical or critical processes.

Assuming that  $g(0) = p_0 > 0$  (otherwise the process never could die out), as the graphical illustration on Fig. 4 shows, there are two distinct cases. As long as  $g'(z)|_{z=1} = \nu \leq 1$ ,  $\nu$  being the average number of male descendants (left hand side of the figure), the only root of the extinction equation (4) is  $q = 1$ , that is the process (the family line) will die out with 100% certainty. If  $\nu > 1$ , i.e. the average number of male descendants is larger than unity, then the extinction equation has two roots, (right hand side of the Figure); one is less than unity ( $q < 1$ ),

the other is the known solution  $q = 1$ . In this latter case, the relevant solution is the smaller root,  $q < 1$ , i.e. the family will not die out with 100% certainty. (The readers who are curious of the proof why  $q < 1$  is the “relevant root”, can find the answer in a recent note (Ref. [8]). This is the root which Watson overlooked.

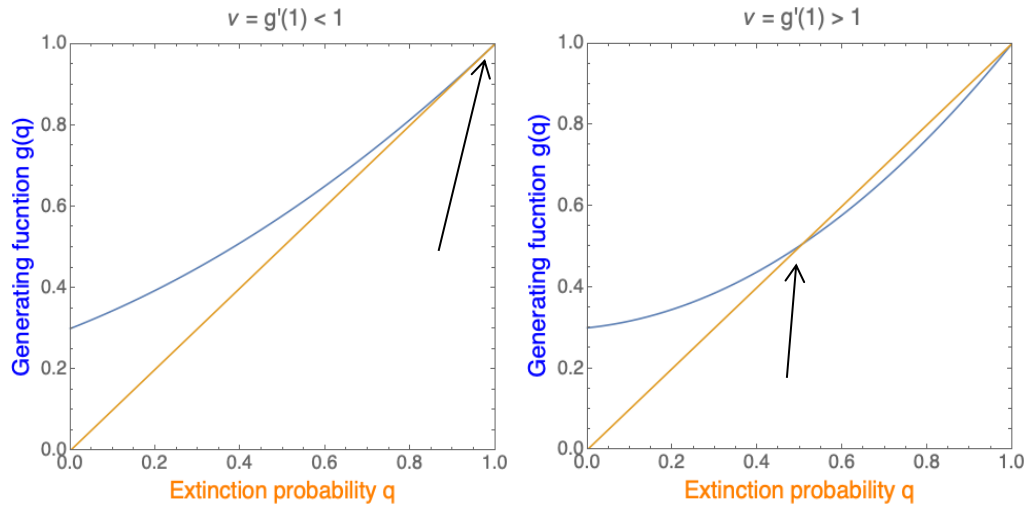


Fig. 4. Solution of the extinction equation

### The extinction and multiplication of neutron chains in a critical reactor

At this point we switch from families to neutrons and interpret the above in terms of neutron chains. The only change is that the male descendants will be replaced by neutrons. The above results say that in a multiplying medium, for  $\nu \leq 1$ , i.e. subcritical and critical systems, a neutron chain started by a single neutron will die out with 100% certainty, whereas for supercritical systems ( $\nu \geq 1$ ), the probability that the chain will die out is less than 100%.

In the continuation, we will focus on exactly critical systems, with  $\nu = 1$ . There, one notices an apparent contradiction. In a critical system the expectation of the number of neutrons is constant. Hence, on the average, a single starting neutron will “live forever”. On the other hand, the above results show that in a critical system, all neutrons will die out with 100% certainty.

But this is not the end of the story yet. It is easy to show (but we abstain from a derivation) that in a critical system, the variance diverges; it tends to infinity linearly with the number of generations (for families) or with time (for neutron chains). This latter result, i.e. that the variance increases linearly with time is also valid to a similar process, the random walk, and in this context it is called the “Einstein relation”. The derivation for neutron chains can be found in [9] and [10].

The divergence of the variance at criticality has long puzzled the nuclear engineering community. This would mean that a reactor, operating at critical, would sooner or later experience a large power excursion. But such a case, a sudden, uncontrolled burst of the neutron flux in a critical reactor, due to purely probabilistic reasons, has never been observed. In Ref. [9] there is a detailed discussion of the possible reasons. The most obvious is that it is the reactor control, not taken into account in the above considerations, which prevents large deviations from the desired flux level. Another suggestion is that a stationary reactor is never critical, only close to critical, because there are always extraneous neutron sources, not arising from the chain reaction, such as cosmic radiation, decay of fission products etc. And as it can also be easily shown, in a subcritical, source-driven reactor (a reactor in which the chain reaction is maintained by an extraneous source), the variance remains finite, and asymptotically constant.

We will show now that, contrary to the standard belief, there is no need to refer to reactor control or extraneous sources to explain why divergence at criticality has not been observed, the explanation lies in the same mathematical theory as the one which creates the apparent contradictory facts. Let us summarize these facts:

In a critical reactor

- The expectation of the number of neutrons is constant;
- The extinction probability equals unity;
- The variance diverges.

How is this possible at the same time?

Here we arrived at a very crucial fact in probability theory which is often overlooked in neutron noise theory, i.e. the concept of ergodicity and its applicability. Namely, the laws and relationships of probability theory are always defined over statistical ensembles, consisting of a large number of identical objects and corresponding events. The laws say nothing about the behaviour of the individual. However, in practice, we have very seldom an ensemble of a large number of identical objects. We do not have thousands of identical reactors to make an experiment; we have only one reactor and would like to extract information on its random behaviour based on this only reactor. Thus, instead of investigating a thousand of identical reactors, we observe our reactor a thousand different times, in the hope that we obtain the same information.

This is the concept of ergodicity, which means that ensemble averages can be replaced by time averages. And the concept sounds plausible. For instance, if we want to investigate if a coin is fair, i.e. whether the probability of heads and tails is equal, we can either toss a thousand identical coins, and count the number of coins with heads and tails, or toss the same coin a thousand times, and count the number of heads and tails. There does not seem to be any difference in the two methods, except that the latter is much easier, because it requires only one coin.

Indeed, in this case there is no difference in the result; and the reason is that the process described above is *ergodic*. In the experiment of tossing the coins and counting the heads and tails, there is no difference between the ensemble average and the time average (average over the number of tossing the same coin).

One would be tempted to think that all physically realistic processes are ergodic, and indeed a great many physical processes are ergodic. But it might come as a surprise how many *are not*; and in particular how a small change in the definition of a process leads to violating ergodicity. Take for instance coin tossing again. But instead of counting the number of heads and tails, we now count the difference between heads and tails, as a function of the number of the tosses. Heads count as +1, and tails as -1, and if there are more tails than heads, the value of the process is negative. This is then a discrete random process, equivalent to a one-dimensional random walk with equal probability of stepping to the left and to the right.

It is easy to prove that this process is not ergodic. A rigorous mathematical proof is not given here, it can easily be found in the literature. One indication that this process is not ergodic is that only stationary processes, whose statistical properties are invariant to a time shift, can be ergodic. For such processes both the mean value and the variance must be constant. But we already mentioned that for a random walk, the variance grows linearly with the number of steps. The constant mean (expectation), together with the diverging variance, is possible through the fact that if we consider a large number of random walks, half of them will diverge to plus infinity, the other half to minus infinity. This gives a mean value of zero. But this requires an ensemble average, i.e. we never get this result if we continue to follow one and the same process. To get the correct result, we must re-start the system.



One can experience the corollaries of these facts in everyday life. For instance this is the reason why a number of ball games, where the difference of the number of scores decides who wins, such as tennis, table tennis, volleyball etc. are played in sets. In tennis, a tournament match goes until one player wins three sets, usually consisting of 6 games won (except in a tie break), instead of one single match consisting of 18 games won. This latter would not be fair. One has to break the “random walk” into pieces, i.e. re-starting the process, to get some ensemble average. One actually made a change about 20 years ago in the rules of table tennis, shifting from 3 sets of 21 points to 4 sets of 11 points, which gives a better ensemble average. The non-ergodicity is part of the reason why a player, winning one set quite comfortably, can lose the second set with equally as large margins.

Now getting back to our critical reactor, the explanation of the three “contradicting” properties above is based on the fact that due to the non-ergodicity of the neutron multiplication in a critical system, it has to be interpreted in terms of ensemble averages, and not time averages, hence they only describe the behaviour of a large number of reactors. It is not the same reactor in which the neutron population will both die out, will remain constant, and diverge at the same time. Rather, if we have a large number of critical reactors, in most of them the process will die out, whereas in a small fraction of the reactors the number of the neutrons will reach a very high value. To get the extinction probability unity, have a constant expectation, and a diverging variance, we will have to take the limit of infinitely many reactors. Then, the fraction of the reactors in which the neutron population does not die out must go to zero, and in these reactors the neutron population must diverge; in the rest, i.e. in almost all reactors, the population will die out. It is easy to confirm that with such a procedure, all three “contradictory” properties can be fulfilled.

The above shows that the expectation is a value which will never be realised; the neutron number dies out in almost every system, and it will diverge in a zero set of all reactors. This also means that the fraction of the reactors in which the flux will diverge is negligible; if we select a reactor at random, it is almost sure that it is not the one which will diverge. In a loose comparison, one could say that the chance that “our” reactor would explode by random fluctuations is comparable with the probability that a person would fall out from an aeroplane by the quantum mechanical tunnel effect. The main point being that the probability is not zero, but sufficiently low that we do not worry about it. According to the above, we should not worry about divergence at critical; we should worry about extinction, since in almost all critical reactors the flux should die out. It is more to these cases that the effects named in Ref. [9], i.e. the control system, the fact that the reactor is slightly subcritical in the presence of extraneous neutrons etc. have to be counted on.

We close this subject with two quotes. The first concerns the answer to the problem statement of Watson, described in the foregoing, i.e. the statistical behaviour of an ensemble of  $N$  family trees. The answer, for the supercritical case ( $\nu \equiv m > 1$ , and hence  $q < 1$ ) is given in a vivid description by Kendall [6] which, although mathematically correct, would find some difficulties to be published in this form today if we read the sentence in italics:

“What this tells us is that if a large number  $N$  of males all having different surnames colonise a district, and if (*females being available as and when required*) they each propagate with a finite average replacement rate  $m > 1$  (with  $\sum k^2 p^k < \infty$ ), then after a long time has elapsed about  $qN$  of the surnames will have disappeared, while the remainder  $(1 - q) N$  will persist forever”.

The second quote reminds to the difference between the statistical behaviour of an ensemble and that of an individual member of the ensemble, especially for non-ergodic processes. This quote is also due to Kendall, but the present author found it in a fascinating book with the title “Modern Mathematics for the Engineer”, published in 1961 [11]. It will not go unnoticed that what was “modern” in 1961 may not sound very modern today, at least regarding computer algorithms, yet I would recommend to each engineering student to have a look at the book, because it really

conforms to the citation, attributed to the famous Hungarian-born mathematician John von Neumann, claiming that “*The purpose of computation is insight, not numbers*”. The section from the book, citing Kendall’s example, is reproduced here with the permission of McGraw-Hill Educational:

### 6. 13 The Example of D. G. Kendall's Taxicab Stand<sup>1</sup>

At a perfectly balanced taxicab stand, either customers wait for taxis or taxis wait for customers. Customers and taxis arrive with equal frequencies. If customers are counted + and taxis -, the queue length may be any integer  $0, \pm 1, \pm 2, \dots$ . From a queue of length  $k$ , the next change leads with equal probabilities to  $k + 1$  or  $k - 1$ . Thus the successive changes are represented by a symmetric random walk. *The expected queue length* is 0, but it is easily seen that at each individual stand the queue length is *bound to grow to  $+\infty$  or  $-\infty$* . The zero expectation says nothing about the fluctuations at an individual stand; it assures us merely that, in a large ensemble, for any stand with thousands of taxis waiting in despair for customers there is somewhere a stand with equally many customers waiting vainly for a taxi.

It should be borne in mind that in building taxi stands, elevators, etc., we are interested in the fluctuations in time at *one particular* counter, not in large ensembles balanced in the manner described. Statistical equilibrium is good where it is really meaningful – e.g., in an ensemble of many telephone trunk lines. But little satisfaction can be derived from a judicial statistical equilibrium where for each innocently condemned person we find a felon running free.

1. Kendall, D. G., Some Problems in the Theory of Queues, *J. Roy. Statist. Soc.*, ser. B, vol. 13, pp. 151-185, 1951.

#### A remark on transport in stochastic media

Kendall’s taxi stand example lends a wonderful illustration and support to the scepticism and reservation of this author concerning the treatment of neutron transport in stochastic media. A stochastic medium is one whose composition (i.e. space-dependence of the cross sections) cannot be described with a deterministic function, rather only through probability distributions or density functions. One practical example of such a medium is the core of a pebble bed reactor, in which neither the exact position of the pebbles in the core during operation, nor the position of the uranium particles inside the pebbles are known. Another example is neutron transport in geological formations, or especially in corium after a core meltdown, where neither the geometry, nor the composition is known.

Mathematically, this is a rather fascinating and challenging area. Not surprisingly, it drew a lot of interest among transport theoreticians. There exists a vast literature of journal publications, and an excellent inventory of the methods and results is found in the book of Pomraning [12]. Many ANS conferences have sessions for “stochastic transport”, which also includes transport in stochastic media.

My problem with this topic is not the academic quality of the work, which is undoubtedly of high standard. Rather, the lack of relevance of the results to any practical situation. Assume for instance, that for a given probability distribution of the random medium, one can show that the expectation of the reactivity for such systems is zero, so that the reactor is “critical on the average”. In the parlance of Kendall’s taxi stand problem, this merely tells us that in a large ensemble of such cores, for a subset of deeply subcritical cores there will be somewhere an equally sized collection of cores that are prompt supercritical. It does not say anything about our reactor, only about the statistical average over a large number of reactors with an identical probability distribution of their core material. This is hardly useful information for a given selected reactor.

The results could be useful if one could do something similar as in the case of stationary ergodic time-dependent process, where one can replace the ensemble average with time average. However, the process here is not time-, but space-dependent (transport in stochastic media is only treated in static problems). Hence, such a step would require replacing ensemble averages with spatial averages. This would require that the system is ergodic in space. But this would only be possible in infinite media (ergodicity requires stationarity), which is irrelevant for practical applications.

The advantage of time-dependent processes (i.e. where the statistics lies in the time evolution of sequence of events), is that time flows from minus infinity to plus infinity. Even in the case when the process is not ergodic, we can estimate ensemble averages by re-starting the process (or its observation, i.e. the measurement). This way one avoids the need of having a large number of identically distributed systems, and useful statistical information can be obtained by observing one and the same system repeatedly. This is the case with the non-ergodic tennis and table tennis games, where the process is restarted after each set. The aggregate of the results of several sets gives an estimate of the ensemble average, without the need of observing a large number of identically skilful players competing simultaneously. Similarly, in the Feynman-alpha or variance to mean method (see the next section), the average and the variance of the number of counts in a given time period is estimated by making several measurements with the same time length. However, there is no such possibility with processes varying randomly in space. Therefore, the results of the corresponding theories, however beautiful and intellectually pleasing they may be, give just as irrelevant information on a given selected reactor core as the expected queue length at a taxi stand.

### **The utilisation of neutron fluctuations in reactors**

How, then, can one utilise the stochastic nature of the neutron population in nuclear engineering? From the reasoning so far, we might conclude that 1) the neutron fluctuations in a reactor (or any system containing fissile material) carry useful, non-trivial information on the system, and 2) we need to be careful how we apply our methods, when we only have access to measurement data from one reactor, and not from an ensemble of identical reactors.

One example where the random character of neutron transport and multiplication is utilised is the Monte Carlo method. Traditionally, this method was invented as a numerical tool to calculate expectations (mean values) of the neutron population, i.e. to obtain a solution of the deterministic neutron transport equation. In the method the fate of a large number of neutrons is followed up, by simulating the possible events (place of next collision, type of reaction, number of secondary neutrons etc.), according to the known probability laws, determined by the material and geometrical properties of the medium, i.e. the reactor core. Ensemble averages are calculated by arranging the simulations into batches, where in each batch the simulations are re-started and repeated a number of times. The advantage of the Monte-Carlo method is that it is very versatile and effective to treat realistic, inhomogeneous systems with complicated geometries, by keeping all variables (space, angle, energy), without making approximations.

However, the Monte-Carlo method is not the truly characteristic utilization of the neutron fluctuations in the spirit described in the previous sections; it is mentioned here only as an illustration of the fact that often, even if only average (“deterministic”) quantities are sought, it often gives advantages to utilize the fact that the underlying process is random. But the Monte-Carlo method does not utilize the information contained in the higher moments of the neutron distribution. The archetype of the applications which does utilize this information is the determination of the subcritical reactivity  $\rho < 0$  of a reactor, in which a stationary neutron population is maintained by an extraneous neutron source. Since in a subcritical system, there is a relationship between the flux level and the subcritical reactivity, one could think that the reactivity  $\rho$  can be determined from the measurement of the number of detector counts  $Z$  during a time period

$t$ . However, for this we should need to know exactly the intensity (strength)  $S$  of the extraneous source, the detector efficiency  $\varepsilon$ , and so on, which are usually not known.

However, since the second moment of a branching process contains independent information from that of the first moment, a more efficient way is to use the relative variance, or variance to mean, of the detector counts  $Z$  as a function of the measurement time  $t$ . In a simplified form, neglecting delayed neutrons, the variance to mean of the detector counts is given by the formula

$$\frac{\sigma_Z^2(t)}{Z(t)} = 1 + Y(t) \equiv 1 + \varepsilon A \left( 1 - \frac{1 - e^{-\alpha t}}{\alpha t} \right); \quad \alpha = \frac{-\rho}{\Lambda} \quad (5)$$

Here,  $A$  is a known nuclear constant, and  $\Lambda$  is the (known) neutron generation time. One can see that the source  $S$  exactly vanishes from the expression, since both the variance and the mean are linearly proportional to it. The detector efficiency  $\varepsilon$  is still present; however, due to the non-linear dependence of the relative variance on the measurement time, the parameter  $\alpha$ , which contains the sought reactivity  $\rho$ , can be determined by curve fitting without the knowledge of  $\varepsilon$ . It is also seen that the relative variance is over-Poisson (larger than unity), and that the sought information is contained in the deviation from the Poisson variance.

The variance to mean method is commonly called the Feynman-alpha method, because it was first suggested, based on heuristic derivations, by R. Feynman and colleagues [13]. The full formula (containing delayed neutrons) was rigorously derived by L. Pál (1925-2019) in his seminal work on the stochastic theory of neutron fluctuations, the so-called Pál-Bell equation [14].

### Concluding remarks

It is the hope of the author that this small essay will induce some interest in the young readers regarding the subtleties and surprises of random processes in general, and neutron fluctuations in particular. This latter area (also called “zero power reactor noise”) is under intensive development both in reactor physics and in nuclear safeguards. The purpose of the latter is to detect, identify and quantify hidden (e.g. smuggled) fissile materials by non-intrusive methods. This is achieved by using auto- and cross moments of neutrons and gamma photons, emitted from the item due to spontaneous fission, up to the third or fourth order. Neutron fluctuations in power reactors, induced by random technological processes (boiling of the coolant, flow-induced vibrations of control rods and fuel assemblies) are used for on-line monitoring of operating power reactors, to detect and identify incipient failures at an early stage. A review of both zero power and power reactor noise is found in Ref. [15]

### REFERENCES

- [1] Ruby, L., Comment on the Physical Interpretation of the Flux. *Nucl Sci Eng* **109**, 111-111 (1991) (Letter to the Editor)
- [2] Ruby, L., Dancoff’s Solution for the Number of Collisions Necessary to Slow Down. *Nucl. Sci. Eng.* **86**, 110-111 (1984) (Technical Note)
- [3] Cohen, E.R., How Many Collisions to Slow Down a Neutron? *Nucl. Sci. Engng* **89**, 99-102 (1985) (Technical Note)
- [4] Dam, H.v., A Difference of at Least Three Collisions. *Nucl. Sci. Eng.* **89**, 365-366 (1985)
- [5] Pázsit, I., Analysis of heartbeat data with spectral and fractal techniques. *Fluct Noise Letters* **5**, L357-L363 (2005).
- [6] Kendall, D.G., Branching Processes Since 1873. *Journal of the London Mathematical Society* s1-41, 385-406 (1966) Address delivered on the occasion of the Centenary of the Society. doi:[10.1112/jlms/s1-41.1.385](https://doi.org/10.1112/jlms/s1-41.1.385)

- [7] Harris, T.E., *The Theory of Branching Processes*. Springer-Verlag, Berlin, Heidelberg (1963)
- [8] Pázsit I., Pál L. and Williams, M.M.R. Note on the extinction probability. Letter to the Editor, *Ann. nucl. Energy* **106**, 271 - 272 (2017)  
doi: 10.1016/j.anucene.2017.04.010
- [9] M.M.R. Williams. *Random Processes in Nuclear Reactors*. Pergamon Press, Oxford (1974)
- [10] I. Pázsit and L. Pál, *Neutron Fluctuations - a Treatise on the Physics of Branching Processes*. Elsevier Ltd., London, New York, Tokyo (2008)
- [11] Beckenbach, E.F., *Modern Mathematics for the Engineer*, McGraw-Hill, New York, 1961
- [12] Pomraning, G.C., *Linear Kinetic Theory and Particle Transport in Stochastic Mixtures*. World Scientific Publishing, River Edge, NJ, USA (1991)
- [13] Feynman, R.P., De Hoffmann, F., Serber, R., 1956. Dispersion of the neutron emission in U-235 fission. *Journal of Nuclear Energy* 3, 64 ((1954)
- [14] Pál, L., On the theory of stochastic processes in nuclear reactors, *Il Nuovo Cimento* Supplemento 7, 25 (1958)
- [15] Pázsit I. and Demazière C., Noise techniques in nuclear systems. In: *Handbook of Nuclear Engineering*. Edited by Cacuci D.G. Vol. 3. New York (NY): Springer; p. 1631–1737 (2010)